2018-11-07-1

Last Time: Numerical integration.

Today: Linear stability

* Numerical solutions are good, but we can only gain intuition on the problem after the simulation has finished running

* Can build up intuition using Linear Stability.

Ex] $\frac{dy}{dt} = \alpha y \qquad y(t=0) = y_0$ (single input system)

$$y = y_0 e^{\alpha t}$$

Let $\alpha = \sigma + j\omega \quad \Rightarrow \quad y = (y_0 e^{\sigma t}) e^{j\omega t}$

$\sigma > 0$	exponential growth: unstable
$\sigma < 0$	exponential decay: stable
$\sigma = 0$	pure oscillations: marginally stable

Ex] $\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} \qquad \underline{x}(t=0) = \underline{x_0}$ (multiple input system)

$$\underline{x} = \exp(\underline{\underline{A}}t)\,\underline{x}_0$$

$$\exp(\underline{\underline{A}}t) = \underline{\underline{I}} + \underline{\underline{A}}t + \frac{1}{2}(\underline{\underline{A}}t)^2 + \dots$$

How to determine stability? Eigenvalues of $\underline{\underline{A}}$ obtained from $\det(\underline{\underline{A}} - \lambda\underline{\underline{I}}) = 0$

$Re\{\lambda\} < 0$ for all λ: stable

$Re\{\lambda\} > 0$ for 1 λ: unstable

Proof: $\underline{\underline{A}}\,\underline{v} = \lambda \underline{v} \quad \Rightarrow \quad \underline{\underline{A}}\,\underline{\underline{V}} = \underline{\underline{V}}\,\underline{\underline{\Lambda}} \qquad \underline{\underline{V}} = [v_1 | v_2 | \dots | v_n] \qquad \underline{\underline{\Lambda}} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \;\lambda_n \end{bmatrix}$

$$\underline{\underline{A}} = \underline{\underline{V}}\,\underline{\underline{\Lambda}}\,\underline{\underline{V}}^{-1} \qquad \underline{\underline{V}}\,\underline{\underline{V}}^{-1} = \underline{\underline{I}}$$

$$\exp(\underline{\underline{A}}t) = \underline{\underline{I}} + (\underline{\underline{V}}\,\underline{\underline{\Lambda}}\,\underline{\underline{V}}^{-1})t + \frac{1}{2}(\underline{\underline{V}}\,\underline{\underline{\Lambda}}\,\underline{\underline{V}}^{-1})^2 t^2 + \dots \Rightarrow \exp(\underline{\underline{A}}t) = \underline{\underline{V}}\,\underline{\underline{I}}\,\underline{\underline{V}}^{-1} + \underline{\underline{V}}\,\underline{\underline{\Lambda}}t\,\underline{\underline{V}}^{-1} + \frac{1}{2}\underline{\underline{V}}(\underline{\underline{\Lambda}}^2 t^2)\underline{\underline{V}}^{-1} + \dots$$

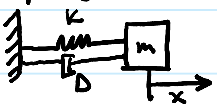
$$\Rightarrow \exp(\underline{\underline{A}}t) = \underline{\underline{V}}\left(\underline{\underline{I}} + \underline{\underline{\Lambda}}t + \frac{1}{2}\underline{\underline{\Lambda}}^2 t^2 + \dots\right)\underline{\underline{V}}^{-1} \Rightarrow \exp(\underline{\underline{A}}t) = \underline{\underline{V}}\left(\begin{bmatrix} \exp(\lambda_1 t) & & 0 \\ & \exp(\lambda_2 t) & \\ 0 & & \ddots \; \exp(\lambda_n t) \end{bmatrix}\right)\underline{\underline{V}}^{-1}$$

$$\Rightarrow \underline{x} = \underline{\underline{V}}\left(\begin{bmatrix} \exp(\lambda_1 t) & 0 & 0 & \dots & 0 \\ 0 & \exp(\lambda_2 t) & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 0 & \exp(\lambda_n t) \end{bmatrix}\right)\underline{\underline{V}}^{-1}\,\underline{x}_0$$

$\underline{x_0}, \underline{\underline{V}}, \underline{\underline{V}}^{-1}$ all constant, so for stability $Re\{\lambda_i\} < 0$ for all i.

2018-11-07-2

Ex. Spring mass damper system



$$m\ddot{x} = -kx - D\dot{x} \Rightarrow \quad \ddot{x} = -\frac{k}{m}x - \frac{D}{m}\dot{x}$$

$$\frac{dx}{dt} = \dot{x}$$

$$\frac{d\dot{x}}{dt} = -\frac{k}{m}x - \frac{D}{m}\dot{x}$$

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{D}{m} \end{bmatrix}}_{\underline{\underline{A}}} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \det\left(\begin{bmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{D}{m} - \lambda \end{bmatrix}\right) = 0$$

$$-\lambda\left(-\frac{D}{m} - \lambda\right) - \left(-\frac{k}{m}\right) = 0$$

$$\lambda^2 + \lambda\frac{D}{m} + \frac{k}{m} = 0$$

$$\lambda = \frac{-\frac{D}{m} \pm \sqrt{\left(\frac{D}{m}\right)^2 - 4\left(\frac{k}{m}\right)}}{2}$$

$$\lambda = \frac{-\frac{D}{m} \pm \sqrt{\frac{1}{m}\left(\frac{D^2}{m} - 4k\right)}}{2}$$

Case 1: $D = 0$

$$\lambda = \pm j\sqrt{\frac{k}{m}}$$

marginally stable

Case 2: $0 < \frac{D^2}{m} < 4k$

$$\lambda = \frac{-\frac{D}{m} \pm j\sqrt{\frac{1}{m}\left(4k - \frac{D^2}{m}\right)}}{2}$$

Stable

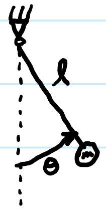
Case 3: $\frac{D^2}{m} > 4k$

$$\lambda = \frac{-\frac{D}{m} \pm \frac{D}{m}\sqrt{1 - \frac{4k}{D^2/m}}}{2}$$

Stable

* All previous examples were linear. What if system of interest is non-linear?

Ex



$m\ell\ddot{\theta} = -mg\sin\theta \Rightarrow \ddot{\theta} = -\frac{g}{\ell}\sin(\theta)$

Equilibrium: $\dot{\theta} = \ddot{\theta} = 0$

$-\frac{g}{\ell}\sin(\theta_{eq}) = 0$

$\boxed{\theta_{eq} = 0, \pi}$

Steps:
1) Find equilibrium points
2) Make equations linear about the equilibrium points (Linearization)
3) Find the Eigenvalues
4) Determine Stability

Linearize about $\theta_{eq} = 0$: $\theta = \theta_{eq} + \Delta\theta \Rightarrow \sin(\theta) = \sin(\theta_{eq} + \Delta\theta)$

Taylor series expand: $\sin(\theta_{eq} + \Delta\theta) = \sin(\theta_{eq}) + \Delta\theta\cos(\theta_{eq}) - \frac{1}{2}\Delta\theta^2\sin(\theta_{eq}) - \frac{1}{3!}\Delta\theta^3\cos(\theta_{eq}) + \dots$

$= 0 + \Delta\theta - 0 - \frac{\Delta\theta^3}{3!} + \dots$

$= \Delta\theta + O(\Delta\theta^3) \approx \Delta\theta$

$\boxed{\Delta\ddot{\theta} = -\frac{g}{\ell}\Delta\theta}$

Linearize about $\theta_{eq} = \pi$: $\theta = \theta_{eq} + \Delta\theta \Rightarrow \sin(\theta) = \sin(\theta_{eq} + \Delta\theta)$

Taylor series expand: $\sin(\theta_{eq} + \Delta\theta) = \sin(\theta_{eq}) + \Delta\theta\cos(\theta_{eq}) - \frac{1}{2}\Delta\theta^2\sin(\theta_{eq}) - \frac{1}{3!}\Delta\theta^3\cos(\theta_{eq}) + \dots$

$= 0 - \Delta\theta + 0 + \frac{\Delta\theta^3}{3!} + \dots$

$= -\Delta\theta + O(\Delta\theta^3) \approx -\Delta\theta$

$\boxed{\Delta\ddot{\theta} = \frac{g}{\ell}\Delta\theta}$

$\theta_{eq}=0$: $\frac{d\Delta\theta}{dt} = \Delta\dot{\theta}$

$\frac{d\Delta\dot{\theta}}{dt} = -\frac{g}{\ell}\Delta\theta$

$$\frac{d}{dt}\begin{bmatrix}\Delta\theta \\ \Delta\dot{\theta}\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1 \\ -\frac{g}{\ell} & 0\end{bmatrix}}_{\underline{\underline{A}}}\begin{bmatrix}\Delta\theta \\ \Delta\dot{\theta}\end{bmatrix}$$

$$\det(\underline{\underline{A}} - \lambda\underline{\underline{I}}) = \det\left(\begin{bmatrix}-\lambda & 1 \\ -\frac{g}{\ell} & -\lambda\end{bmatrix}\right) = 0$$

$$\lambda^2 - \left(-\frac{g}{\ell}\right) = 0$$

$$\lambda^2 + \frac{g}{\ell} = 0$$

$$\boxed{\lambda = \pm j\sqrt{\frac{g}{\ell}}}$$

$\underline{\theta_{eq} = \pi}$: $\frac{d\Delta\theta}{dt} = \Delta\dot{\theta}$

$\frac{d\Delta\dot{\theta}}{dt} = \frac{g}{\ell}\Delta\theta$

$$\frac{d}{dt}\begin{bmatrix}\Delta\theta \\ \Delta\dot{\theta}\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1 \\ \frac{g}{\ell} & 0\end{bmatrix}}_{\underline{\underline{A}}}$$

$$\det(\underline{\underline{A}} - \lambda\underline{\underline{I}}) = \det\left(\begin{bmatrix}-\lambda & 1 \\ \frac{g}{\ell} & -\lambda\end{bmatrix}\right) = 0$$

$$\lambda^2 - \frac{g}{\ell} = 0$$

$$\lambda^2 = \frac{g}{\ell}$$

$$\boxed{\lambda = \pm\sqrt{\frac{g}{\ell}}}$$

$\theta_{eq}=0$: marginally stable
$\theta_{eq}=\pi$: unstable